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A PROBLEM IN THERMAL-CONDUCTIVITY THEORY

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A solution of the problem of heat distribution in an infinite lamina is presented.

Let an infinite lamina of thickness $2l(-l \leq x \leq l)$ have an initial temperature u_0. Over the course of a time t_1 it is heated by a constant thermal flux of density q, as a result of which the temperature of the surfaces bounding the lamina becomes equal to u_1. It is required to determine by what law the thermal flux must change further in order that the lamina surfaces be maintained at this temperature u_1. Initially, we find the temperature distribution law at the end of heating, i.e., after expiration of time t_1. To do this we use the solution of the thermal-conductivity equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{1}$$

with initial condition

$$u(x,\ 0) = u_0 \tag{2}$$

and boundary conditions

$$-\lambda \frac{\partial u(x,\ t)}{\partial x}\bigg|_{x=-l} = q, \tag{3}$$

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$$\lambda \left.\frac{\partial u(x,\ t)}{\partial x}\right|_{x=l} = q. \tag{4}$$

This solution is given by the formula [2]

$$u_1(x,\ t) = u_0 + \frac{q}{\lambda} \cdot \frac{1}{l}\int_0^t \vartheta_3\left(\frac{l-x}{2l},\ \frac{\tau}{l^2}\right) d\tau, \tag{5}$$

where x is the coordinate of the point at which temperature is measured, and the theta function ϑ_3 is defined by

$$\vartheta_3(v,\ x) = 1 + 2\sum_{k=1}^{\infty} \exp(-\pi^2 k^2 x)\cos 2\pi k v. \tag{6}$$

We will find the temperature on the lamina boundary surfaces (at $x = -l$ and $x = l$). By means of a Laplace transform, we transform the integral of Eq. (5) to the following form:

$$\frac{1}{l}\int_0^t \vartheta_3\left(\frac{l-x}{2l},\ \frac{\tau}{l^2}\right) d\tau \doteqdot \frac{\operatorname{ch}(x\sqrt{p})}{\sqrt{p}\operatorname{sh}(l\sqrt{p})}$$

$$= \frac{1}{\sqrt{p}}\left[\frac{\exp(x\sqrt{p}) + \exp(-x\sqrt{p})}{\exp(l\sqrt{p}) - \exp(-l\sqrt{p})}\right] = \frac{1}{\sqrt{p}}\sum_{n=1}^{\infty}\exp\{-[(2n-1)l-x]\sqrt{p}\}$$

$$+ \frac{1}{\sqrt{p}}\sum_{n=1}^{\infty}\exp\{-[(2n-1)l+x]\sqrt{p}\}. \tag{7}$$

From the numerical calculation formula

$$\frac{1}{\sqrt{p}}\exp(-\alpha\sqrt{p}) \doteqdot 2\sqrt{t}\operatorname{ierfc}\left(\frac{\alpha}{2\sqrt{t}}\right), \tag{8}$$

where the integral error function is defined by

$$\operatorname{ierfc} w = \int_w^{\infty} \operatorname{erfc}\xi\, d\xi, \tag{9}$$

we obtain from Eq. (7)

$$\frac{1}{l}\int_0^t \vartheta_3\left(\frac{l-x}{2l},\ \frac{\tau}{l^2}\right) d\tau = 2\sqrt{t}\sum_{n=1}^{\infty}\left[\operatorname{ierfc}\frac{(2n-1)l-x}{2\sqrt{t}} + \operatorname{ierfc}\frac{(2n-1)l+x}{2\sqrt{t}}\right]. \tag{10}$$

Thus,

$$u_1(x,\ t) = u_0 + \frac{q}{\lambda}\, 2\sqrt{t}\sum_{n=1}^{\infty}\left[\operatorname{ierfc}\frac{(2n-1)l-x}{2\sqrt{t}} + \operatorname{ierfc}\frac{(2n-1)l+x}{2\sqrt{t}}\right]. \tag{11}$$

Equation (11) permits determination of the time over which heating must be continued to obtain the required temperature on the lamina surfaces.

We have from Eq. (11)

$$u_1(-l,\ t) = u_1(l,\ t) = u_1 = u_0 + \frac{q}{\lambda}\, 2\sqrt{t}\sum_{n=1}^{\infty}\left[\operatorname{ierfc}\frac{(n-l)1}{\sqrt{t}} + \operatorname{ierfc}\frac{nl}{\sqrt{t}}\right]$$

$$= u_0 + \frac{q}{\lambda}\, 2\sqrt{t}\left[\operatorname{ierfc} 0 + 2\sum_{n=1}^{\infty}\operatorname{ierfc}\frac{nl}{\sqrt{t}}\right] = u_0 + \frac{q}{\lambda}\sqrt{t}\, 2\operatorname{ierfc} 0$$

$$+ 4\frac{q}{\lambda}\sqrt{t}\sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{\pi}}\exp\left(-\frac{n^2 l^2}{t}\right) - \frac{nl}{\sqrt{t}}\operatorname{erfc}\frac{nl}{\sqrt{t}}\right]. \tag{12}$$

Using an asymptotic expansion of the error function $nl/\sqrt{t}$ for large $w = nl/\sqrt{t}$

$$\operatorname{erfc} w \sim \frac{2}{\sqrt{\pi}} \frac{\exp(-w^2)}{2w}, \tag{13}$$

from Eq. (12) we obtain

$$u_1(-l, t) = u_1(l, t) \approx u_0 + \frac{q}{\lambda} 2 \operatorname{ierfc} 0 \sqrt{t}. \tag{14}$$

Consequently, the desired time is given by the formula

$$t_1 \approx \left[\frac{(u_1 - u_0)\lambda}{2q \operatorname{ierfc} 0}\right]^2, \tag{15}$$

where [5] $2\operatorname{ierfc} 0 = 1.1284$.

In order to find the law for thermal flux change necessary for maintenance of the constant temperature u_1 on the lamina surface (at $x = -l$ and $x = l$), we consider an additional problem. We find the solution of thermal-conductivity Eq. (1) satisfying the initial condition (see [11])

$$u(x, 0) = K_1(x) = u_0 + \frac{q}{\lambda} 2\sqrt{t_1} \sum_{n=1}^{\infty} \left[\operatorname{ierfc} \frac{(2n-1)l - x}{2\sqrt{t_1}} + \operatorname{ierfc} \frac{(2n-1)l + x}{2\sqrt{t_1}}\right] \tag{16}$$

and boundary conditions

$$u(-l, t) = u(l, t) = u_1, \tag{17}$$

where t_1 is determined by Eq. (15) and t is the time passed since the end of heating. As is well known [2], such a solution is given by the formula

$$u(x, t) = u_1 \frac{1}{l} \int_0^t \frac{\partial \vartheta_2\left(\frac{l-x}{2l}, \frac{\tau}{l^2}\right)}{\partial x} d\tau + \frac{1}{2l} \int_0^l \left[\vartheta_2\left(\frac{x-\xi}{2l}, \frac{t}{l^2}\right) + \vartheta_2\left(\frac{x+\xi}{2l}, \frac{t}{l^2}\right)\right] K_1(\xi)\, d\xi, \tag{18}$$

where the theta function ϑ_2 is defined by

$$\vartheta_2(v, x) = 2 \sum_{k=0}^{\infty} \exp\left[-\pi^2 \left(k + \frac{1}{2}\right)^2 x\right] \cos[\pi(2k+1) v]. \tag{19}$$

With the aid of operational calculation the first integral of Eq. (18) may be reduced to the form [3]

$$u_1 \frac{1}{l} \int_0^t \frac{\partial}{\partial x} \vartheta_2\left(\frac{l-x}{2l}, \frac{\tau}{l^2}\right) d\tau \doteqdot u_1 \frac{1}{l} \cdot \frac{1}{p} \cdot \frac{lp \operatorname{ch}(x\sqrt{p})}{\operatorname{ch}(l\sqrt{p})}$$

$$\doteqdot u_1 \left\{1 + \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1} \exp\left[-t\left(k - \frac{1}{2}\right)^2 \frac{\pi^2}{l^2}\right] \cos \frac{\pi x (2k-1)}{2l}\right\}. \tag{20}$$

Knowing temperature distribution (18), we can find the heat flux on the lamina surfaces from the formula

$$q(t) = -\lambda \left.\frac{\partial u(x, t)}{\partial x}\right|_{x=-l} = \lambda \left.\frac{\partial u(x, t)}{\partial x}\right|_{x=l}. \tag{21}$$

Differentiating Eq. (18) with respect to x and considering Eq. (20) and the relationships

$$\frac{\partial}{\partial x} \vartheta_2\left(\frac{x-\xi}{2l}, \frac{t}{l^2}\right) = -\frac{\partial}{\partial \xi} \vartheta_2\left(\frac{x-\xi}{2l}, \frac{t}{l^2}\right),$$

$$\frac{\partial}{\partial x} \vartheta_2\left(\frac{x+\xi}{2l}, \frac{t}{l^2}\right) = \frac{\partial}{\partial \xi} \vartheta_2\left(\frac{x+\xi}{2l}, \frac{t}{l^2}\right),$$

we obtain

$$\frac{\partial u(x, t)}{\partial x} = \frac{u_1}{l} \vartheta_1\left(\frac{x}{2l}, \frac{t}{l^2}\right) - \frac{1}{2l} \int_0^l \frac{\partial}{\partial \xi}\left[\vartheta_2\left(\frac{x-\xi}{2l}, \frac{t}{l^2}\right) - \vartheta_2\left(\frac{x+\xi}{2l}, \frac{t}{l_2}\right)\right] K_1(\xi)\, d\xi, \tag{22}$$

where the theta function ϑ_1 is defined by

$$\vartheta_1(v,\ x) = 2\sum_{k=0}^{\infty}(-1)^k \exp\left[-\pi^2\left(k+\frac{1}{2}\right)^2 x\right]\sin\pi(2k+1)v. \tag{23}$$

Integrating by parts, we obtain

$$\frac{\partial u(x,\ t)}{\partial x} = \frac{u_1}{l}\vartheta_1\left(\frac{x}{2l},\ \frac{t}{l^2}\right) - \frac{1}{2l}\left[\vartheta_2\left(\frac{x-\xi}{2l},\ \frac{t}{l^2}\right)\right.$$

$$\left.-\vartheta_2\left(\frac{x+\xi}{2l},\ \frac{t}{l^2}\right)\right]K_1(\xi)\Big|_{\xi=0}^{\xi=l} + \frac{1}{2l}\int_0^l\left[\vartheta_2\left(\frac{x-\xi}{2l},\ \frac{t}{l^2}\right) - \vartheta_2\left(\frac{x+\xi}{2l},\ \frac{t}{l^2}\right)\right]K_1'(\xi)\,d\xi. \tag{24}$$

Considering the periodicity of the theta function $\vartheta_2(v+1,x) = -\vartheta_2(v,x)$ and the fact that $K_1(l) = u_1$, we find

$$\frac{\partial u(x,\ t)}{\partial x} = \frac{u_1}{l}\vartheta_1\left(\frac{x}{2l},\ \frac{t}{l^2}\right) + \frac{u_1}{l}\vartheta_2\left(\frac{x+l}{2l},\ \frac{t}{l^2}\right) + \frac{1}{2l}\int_0^l\left[\vartheta_2\left(\frac{x-\xi}{2l},\ \frac{t}{l^2}\right) - \vartheta_2\left(\frac{x+\xi}{2l},\ \frac{t}{l^2}\right)\right]K_1'(\xi)\,d\xi. \tag{25}$$

Differentiating Eq. (16), we find

$$K_1'(x) = \frac{q}{\lambda}\sum_{n=1}^{\infty}\left[\operatorname{erfc}\frac{(2n-1)l-x}{2\sqrt{t_1}} - \operatorname{erfc}\frac{(2n-1)l+x}{2\sqrt{t_1}}\right] = \frac{q}{\lambda}\left[\frac{x}{l} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\exp\left(-\frac{k^2\pi^2 t_1}{l^2}\right)\sin\frac{k\pi x}{l}\right]. \tag{26}$$

We use the formula

$$\frac{d}{dw}(i^n \operatorname{erfc} w) = -i^{n-1}\operatorname{erfc} w.$$

In view of Eqs. (21), (25), (26) and considering the equation

$$\vartheta_2(v,\ x) = \vartheta_1\left(\frac{1}{2} - v,\ x\right),\ \vartheta_1(v+1,\ x) = -\vartheta_1(v,\ x),$$

$$\vartheta_2(v+1,\ x) = -\vartheta_2(v,\ x),$$

we obtain

$$q(t) = \lambda\frac{1}{2l}\int_0^l\left[\vartheta_2\left(\frac{x-\xi}{2l},\ \frac{t}{l^2}\right) - \vartheta_2\left(\frac{x+\xi}{2l},\ \frac{t}{l^2}\right)\right]K_1'(\xi)\,d\xi = \frac{\lambda}{l}\int_0^l\vartheta_2\left(\frac{x+\xi}{2l},\ \frac{t}{l^2}\right)K_1'(\xi)\,d\xi,$$

where $K_1'(x)$ is determined from Eq. (26). In performing practical calculations one may limit the examination to the first terms of the series in Eq. (26).

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